

ANALYTIC FUNCTIONS WITH QUASI-ANALYTIC BOUNDARY VALUES⁽¹⁾

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1. Introduction. Given a sequence of positive numbers $A_1, A_2, \dots$, let $F(A_n)$ be the set of functions $f = \sum a_n z^n$ analytic in the disc $D = \{z : |z| < 1\}$ such that

- (i) $f^{(k)}$ has a continuous extension to D^- for $k=0, 1, \dots$,
- (ii) $\{a_n A_n\}$ is a bounded sequence.

We call $F(A_n)$ *quasi-analytic* if $f \in F(A_n)$ and $f^{(k)}(1)=0$ ($k=0, 1, \dots$) together imply that $f \equiv 0$. The main purpose of this paper is to give a necessary and sufficient condition on the sequence $\{A_n\}$ for $F(A_n)$ to be quasi-analytic (Theorem 3.4) and to extend the results to functions analytic in the half-plane and functions of several variables.

In §2 we prove a quasi-analytic theorem on a related class of functions. This theorem is stated without proof by B. I. Korenbljum in [3] and [4]. §3 is devoted to the main Theorem 3.4: If $\log A_n$ is convex in $\log n$, then $F(A_n)$ is quasi-analytic if and only if $\sum_1^\infty n^{-3/2} \log A_n$ diverges. Half of Theorem 3.4 is proven by a slightly different method in [2, p. 331]. There Carleson shows that if $\log A_n$ is convex in $\log n$ and $\sum_1^\infty n^{-3/2} \log A_n$ diverges then functions in $F(A_n)$ can have only finitely many zeros. The first step of his proof reduces this problem to showing that $F(A_n)$ must be quasi-analytic.

In §4 we give a half-plane analogue of Theorem 3.4 by looking at functions of the form $f(z) = \int_0^\infty \hat{f}(t) e^{itz} dt$, where $\hat{f}(t)$ is a measurable function and $\hat{f}(t)A(t)$ is bounded. In §5 we give one possible analogue of Theorem 3.4 to functions of several variables. This result is obtained by applying the one variable theorem.

2. Functions with bounded derivatives. Since we make use of the following computation throughout this paper, we state it here in the form of a lemma.

LEMMA 2.1. *Suppose that f is analytic in the unit disc D , $f^{(k)}$ has a continuous extension to D^- and $f^{(k)}(1)=0$ for $k=0, 1, \dots, n-1$. Then for $k=0, 1, \dots, n-1$ and z_0 in D^- ,*

$$f^{(k)}(z_0) = \frac{1}{(n-k-1)!} \int_1^{z_0} (z_0 - z)^{n-k-1} f^{(n)}(z) dz.$$

Proof. Write $f^{(k)}(z_0) = \int_1^{z_0} f^{(k+1)}(z) dz$ and integrate by parts.

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Given a sequence $\{M_n\}$ of positive numbers, let $C(M_n)$ be the set of functions f analytic in the disc $D = \{z : |z| < 1\}$ such that for $n = 0, 1, \dots$,

- (i) $f^{(n)}$ has a continuous extension to D^- ;
- (ii) $|f^{(n)}(z)| \leq BM_n$ in D for some number $B = B(f)$.

The class $C(M_n)$ is said to be *quasi-analytic* if $f \in C(M_n)$ and $f^{(n)}(1) = 0$ ($n = 0, 1, \dots$) together imply that $f \equiv 0$.

THEOREM 2.2 (KORENBLJUM). *Let $T(r) = \sup_{n \geq 0} r^n M_n^{-1}$. The class $C(M_n)$ is quasi-analytic if and only if $\int_1^\infty r^{-3/2} \log T(r) dr$ diverges.*

Proof. Suppose that the integral converges. Then by Theorem 3 of [3] there is a function $F(w) \not\equiv 0$ analytic in the right half-plane H such that $F^{(n)}$ has a continuous extension to H^- , $|F^{(n)}(w)| \leq M_n$ in H and $F^{(n)}(0) = 0$ for $n = 0, 1, \dots$. Let $f(z) = F(1-z)$. Then f is in $C(M_n)$ and $f^{(n)}(1) = 0$ for $n = 0, 1, \dots$; but $f \not\equiv 0$. Hence $C(M_n)$ is not quasi-analytic.

Conversely, suppose that the integral diverges and f is a function in $C(M_n)$ such that $f^{(n)}(1) = 0$, $n = 0, 1, \dots$. Computing the derivatives of $F(w) = f((1-w)/(1+w))$ gives an expression of the form

$$F^{(n)}(w) = \sum_{k=1}^n a_{nk} (1+w)^{-(n+k)} f^{(k)}\left(\frac{1-w}{1+w}\right)$$

where the a_{nk} are real numbers. Differentiating this expression and comparing coefficients, we obtain the formula

$$\begin{aligned} a_{n+1k} &= -(n+k)a_{nk} - 2a_{nk-1}, \\ a_{n0} &= a_{nn+1} = 0, \quad |a_{nn}| = 2^n. \end{aligned}$$

Using the binomial expansion

$$(n+1)^{2(n+1-k)} = n^{2(n+1-k)} + 2(n+1-k)n^{2(n+1-k)-1} + \dots,$$

an induction argument on n shows that $|a_{nk}| \leq 2^n n^{2(n-k)}/(n-k)!$. For $k = 0, \dots, n-1$ we have

$$f^{(k)}(z_0) = \frac{1}{(n-k-1)!} \int_1^{z_0} (z_0 - z)^{n-k-1} f^{(n)}(z) dz,$$

by Lemma 2.1. Hence

$$|f^{(k)}(z_0)| \leq \frac{1}{(n-k-1)!} 2^{n-k} BM_n.$$

And

$$\begin{aligned} |F^{(n)}(w)| &\leq \sum_{k=1}^n \frac{a_{nk}}{|1+w|^{n+k}} \left| f^{(k)}\left(\frac{1-w}{1+w}\right) \right| \\ &\leq \sum_{k=1}^{n-1} 2^n \frac{n^{2(n-k)}}{(n-k)!} \frac{2^{n-k}}{(n-k-1)!} BM_n + 2^n BM_n. \end{aligned}$$

Now the first term is bounded by

$$4^n BM_n \sum_{k=1}^{n-1} \frac{n^{2(n-k)}}{[(n-k-1)!]^2} = 4^n BM_n \sum_{k=0}^{n-2} \frac{n^{2k+2}}{(k!)^2}.$$

So we conclude that

$$|F^{(n)}(w)| \leq 4^n BM_n n^2 \sum_{k=0}^n \left(\frac{n^k}{k!}\right)^2 \leq 4^n BM_n n^2 (e^n)^2.$$

Again referring to Theorem 3 of [3] we conclude that $F \equiv 0$ and hence $f \equiv 0$.

We define a class of functions similar to $C(M_n)$ as follows. If f is analytic in D and has a continuous extension to D^- , denote the boundary values by $\tilde{f}(\theta) = f(e^{i\theta})$. Let $\tilde{C}(M_n)$ be the set of functions analytic on D such that for each $n=0, 1, \dots$,

(i) $f^{(n)}$ has a continuous extension to D^- ;

(ii) $|\tilde{f}^{(n)}(\theta)| \leq BM_n$ on $[0, 2\pi]$.

The class $\tilde{C}(M_n)$ is said to be *quasi-analytic* if $f \in \tilde{C}(M_n)$ and $\tilde{f}^{(n)}(0) = 0$ ($n=0, 1, \dots$) together imply that $f \equiv 0$. Estimates similar to those in the proof of Theorem 2.2 show that $\tilde{C}(M_n)$ is quasi-analytic if and only if $C(M_n)$ is.

3. Functions with small Taylor coefficients. Mandelbrojt [6] has shown that if $g(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ is infinitely differentiable, $\{a_n A_{|n|}\}_{n=0}^{\infty}$ is bounded, and $g^{(n)}(0) = 0$ ($n=0, 1, \dots$), then a necessary and sufficient condition to conclude that $g \equiv 0$ is that $\sum_{n=1}^{\infty} n^{-2} \log A_n = \infty$. The main theorem of this section, Theorem 3.4, says that if $g(\theta)$ comes from the boundary values of an analytic function, then the appropriate condition is that $\sum_{n=1}^{\infty} n^{-3/2} \log A_n = \infty$. Thus by making an additional assumption on the functions, we can relax the condition on the sequence $\{A_n\}$. The methods we use are much the same as the methods Mandelbrojt uses.

In this section, $A_1, A_2, \dots$ will be a sequence of positive numbers such that $\log A_n$ is convex in $\log n$, i.e.

$$s_n = \frac{\log A_{n+1} - \log A_n}{\log(n+1) - \log n}$$

is a nondecreasing sequence. We let $F(A_n)$ be the set of functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in D such that $f^{(n)}(z)$ has a continuous extension to D^- ($n=0, 1, \dots$) and such that $\{A_k a_k\}$ is bounded. We call $F(A_n)$ quasi-analytic if $f \in F(A_n)$ and $f^{(n)}(1) = 0$ ($n=0, 1, \dots$) together imply that $f \equiv 0$.

We begin by proving some properties of the sequence $\{A_n\}$.

LEMMA 3.1. *If $\lim_{n \rightarrow \infty} s_n = \infty$, then for each nonnegative integer k , $\{n^k A_n^{-1}\}$ is a bounded sequence.*

Proof. Fix k and choose n_0 so that $s_{n_0} \geq k$. Then for $n \geq n_0$,

$$\begin{aligned} \log A_{n+1} - \log A_n &= s_n (\log(n+1) - \log n) \\ &\geq k (\log(n+1) - \log n). \end{aligned}$$

Hence $n^k A_n^{-1} \geq (n+1)^k A_{n+1}^{-1}$; and $n^k A_n^{-1}$ is nonincreasing for $n \geq n_0$.

This lemma allows us to state the following proposition which is crucial in relating $F(A_n)$ to a class of functions of the type $\tilde{C}(M_n)$.

PROPOSITION 3.2. *Suppose that $s_n \geq 0$ and $\lim_{n \rightarrow \infty} s_n = \infty$. Let $c(n) = \sup_{k \geq 1} k^n A_k^{-1}$ and let $d(t) = \sup_{n \geq 1} t^n c(n)^{-1}$. Then for $n = 1, 2, \dots$, $d(n) \leq A_n \leq nd(n)$. Furthermore, given n_0 , there exists a number t_{n_0} such that for $t \geq t_{n_0}$, $d(t) = \sup_{n \geq n_0} t^n c(n)^{-1}$.*

Proof. By the previous lemma $c(n)$ is finite, so that the definition of $d(t)$ is meaningful. For fixed n and for each k , $A_n \geq n^k c(k)^{-1}$. Hence $A_n \geq d(n)$.

To prove the second inequality, fix n and let k be the integer such that $k \leq s_n < k+1$. An argument similar to the proof of Lemma 3.1 shows that for $j \geq n$,

$$j^k A_j^{-1} \leq n^k A_n^{-1} < n^{k+1} A_n^{-1}.$$

While, for $1 \leq j < n$,

$$\begin{aligned} \log A_{j+1} - \log A_j &= s_j(\log(j+1) - \log j) \\ &< (k+1)(\log(j+1) - \log j). \end{aligned}$$

So that

$$j^{k+1} A_j^{-1} < (j+1)^{k+1} A_{j+1}^{-1} \leq n^{k+1} A_n^{-1}.$$

We therefore have $c(k) = \sup_{j \geq 1} j^k A_j^{-1} \leq n^{k+1} A_n^{-1}$. So that $A_n \leq n^{k+1} c(k)^{-1} \leq nd(n)$.

Finally, given n_0 choose $t_{n_0} \geq 1$ so that $t_{n_0} c(n) > c(n_0)$ for each n less than n_0 . Now fix $t \geq t_{n_0}$. For $n < n_0$,

$$t^{n_0} c(n) \geq t^{n+1} c(n) \geq t^n t_{n_0} c(n) > t^n c(n_0).$$

Thus $t^{n_0} c(n_0)^{-1} > t^n c(n)^{-1}$. And hence for $t \geq t_{n_0}$,

$$d(t) = \sup_{n \geq 0} t^n c(n)^{-1} = \sup_{n \geq n_0} t^n c(n)^{-1}.$$

Before proving the main theorem of this section, we prove the following special case which contains the crux of the argument.

THEOREM 3.3. *Suppose that $s_n \geq 0$ and $\lim_{n \rightarrow \infty} s_n = \infty$. Then a necessary and sufficient condition for $F(A_n)$ to be quasi-analytic is that*

$$(1) \quad \sum_{n=1}^{\infty} n^{-3/2} \log A_n$$

diverge.

Proof. Sufficiency. We first show that $F(A_n)$ is contained in $\tilde{C}(M_n)$ where $M_n = c(n+2)$. Suppose f is in $F(A_n)$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $|a_n| \leq B A_n^{-1}$ for some constant B . Now $\tilde{f}(\theta) = \sum_{n=0}^{\infty} a_n e^{i n \theta}$. So for $k = 1, 2, \dots$,

$$\begin{aligned} |\tilde{f}^{(k)}(\theta)| &= \left| \sum_{n=0}^{\infty} a_n (in)^k e^{i n \theta} \right| \leq \sum_{n=1}^{\infty} |a_n| n^k \\ &\leq \sum_{n=1}^{\infty} n^{-2} n^{k+2} B A_n^{-1} \leq B c(k+2) \sum_{n=1}^{\infty} n^{-2}. \end{aligned}$$

Hence $f \in \tilde{C}(M_n)$ where $M_n = c(n+2)$.

Therefore it is enough to show that the divergence of (1) implies that $\tilde{C}(M_n)$ is quasi-analytic, i.e. that $\int_1^\infty r^{-3/2} \log T(r)$ diverges. Now

$$T(r) = \sup_{k \geq 0} r^k c(k+2)^{-1} = r^{-2} \sup_{k \geq 0} r^{k+2} c(k+2).$$

So for $r \geq t_2$, $T(r) = r^{-2} d(r)$. By Proposition 3.2 the divergence of (1) is equivalent to the divergence of $\sum_1^\infty n^{-3/2} \log d(n)$ and hence to the divergence of

$$\int_1^\infty r^{-3/2} \log T(r).$$

Necessity. Let $M_0 = 1$ and $M_k = c(k-1)$ for $k = 1, 2, \dots$. We first show that the convergence of (1) implies that $\tilde{C}(M_n)$ is not quasi-analytic. Let $T(r) = \sup_{k \geq 0} r^k M_k^{-1}$. Then for $r \geq M_1$,

$$T(r) = \sup_{k \geq 0} r^k M_k^{-1} = r \sup_{k \geq 1} r^{k-1} c(k-1)^{-1} = rd(r).$$

By an argument similar to the above, the convergence of the series (1) implies that $\int_1^\infty r^{-3/2} \log T(r)$ converges and therefore $\tilde{C}(M_n)$ is not quasi-analytic.

Hence there is a function f in $\tilde{C}(M_n)$ such that $\tilde{f}^{(n)}(0) = 0$ ($n = 0, 1, \dots$) but $f \not\equiv 0$. The proof of the theorem will be complete if we show that f is also in $F(A_n)$. Fix $n \geq 1$. Integrating by parts we find that

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(\theta) e^{-in\theta} d\theta = \dots = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}^{(n)}(\theta) \left(\frac{1}{in}\right)^k e^{-in\theta} d\theta.$$

And

$$|a_n| \leq BM_k \left(\frac{1}{n}\right)^k \leq \frac{B}{n} \frac{1}{d(n)} \leq \frac{B}{A_n}.$$

Therefore $\{|a_n|A_n\}$ is bounded and f is in $F(A_n)$.

Theorem 3.3 can be improved to give the main result of this section.

THEOREM 3.4. *If $\log A_n$ is a convex sequence in $\log n$, then a necessary and sufficient condition for $F(A_n)$ to be quasi-analytic is that*

$$(1) \quad \sum_{n=1}^{\infty} n^{-3/2} \log A_n$$

diverge.

Proof. First suppose that $\lim_{n \rightarrow \infty} s_n = \infty$. If $s_n \geq 0$ ($n = 1, 2, \dots$), then Theorem 3.3 applies. If not, then since $\lim_{n \rightarrow \infty} s_n = \infty$, there is an integer n_0 such that $s_n \geq 0$ for all $n \geq n_0$. Let

$$\begin{aligned} \bar{A}_n &= A_{n_0} & \text{if } n \leq n_0, \\ &= A_n & \text{if } n \geq n_0. \end{aligned}$$

Then

$$\begin{aligned} \bar{s}_n &= 0 & \text{if } n < n_0, \\ &= s_n & \text{if } n \geq n_0, \end{aligned}$$

where

$$\bar{s}_n = \frac{\log \bar{A}_{n+1} - \log \bar{A}_n}{\log (n+1) - \log n}.$$

Hence $\log \bar{A}_n$ is convex in $\log n$, $\bar{s}_n \geq 0$ and $\lim_{n \rightarrow \infty} \bar{s}_n = \infty$. So Theorem 3.3 applies to $F(\bar{A}_n)$. But $F(\bar{A}_n) = F(A_n)$ and $\sum_1^\infty n^{-3/2} \log \bar{A}_n$ diverges if and only if (1) diverges. Hence the theorem is true whenever $\lim_{n \rightarrow \infty} s_n = \infty$.

Now suppose that it is not the case that $\lim_{n \rightarrow \infty} s_n = \infty$. Then, since $\{s_n\}$ is a nondecreasing sequence, we must have $s_n \leq b$ for some number b . In this case we show (1) converges and $F(A_n)$ is not quasi-analytic. Since

$$\begin{aligned} \log A_{n+1} - \log A_n &= s_n(\log (n+1) - \log n) \\ &\leq b(\log (n+1) - \log n), \end{aligned}$$

we have

$$A_{n+1} \leq A_n \left(\frac{n+1}{n} \right)^b \leq A_{n-1} \left(\frac{n}{n-1} \right)^b \left(\frac{n+1}{n} \right)^b \leq \cdots \leq A_1 (n+1)^b.$$

And

$$\sum_1^\infty n^{-3/2} \log A_n \leq \sum_1^\infty n^{-3/2} (\log A_1 + b \log n) < \infty.$$

To show $F(A_n)$ is not quasi-analytic we exhibit a non-quasi-analytic class $F(\bar{A}_n)$ which is contained in $F(A_n)$. Let $\bar{A}_n = A_1 n^{b \log n}$. Since $A_n \leq A_1 n^b \leq \bar{A}_n$ for $n \geq 3$, $F(\bar{A}_n)$ is contained in $F(A_n)$. And since

$$\bar{s}_n = \frac{\log \bar{A}_{n+1} - \log \bar{A}_n}{\log (n+1) - \log n} = b(\log (n+1) + \log n),$$

we may apply Theorem 3.3 to $F(\bar{A}_n)$. But

$$\sum_1^\infty n^{-3/2} \log \bar{A}_n = \sum_1^\infty n^{-3/2} (\log A_1 + b(\log n)^2) < \infty.$$

So $F(\bar{A}_n)$ is not quasi-analytic.

A number of comments on Theorem 3.4 should be made. The first is that if the series (1) diverges, then any function $f(z) = \sum_0^\infty a_k z^k$, such that $\{a_k A_k\}$ is bounded, is already in $F(A_n)$, i.e. $f^{(n)}(z)$ has a continuous extension to D^- for $n=0, 1, \dots$. For as we have seen, the divergence of (1) implies that $\lim_{n \rightarrow \infty} s_n = \infty$. Thus, by Lemma 3.1, $\{n^{k+2} A_n^{-1}\}$ is bounded for each k . Hence $f^{(k)}(z) = \sum_{n=k}^\infty a_n z^{n-k} n! / (n-k)!$ converges uniformly on D^- , since

$$\left| \frac{n!}{(n-k)!} a_n \right| \leq n^k |a_n| \leq n^{-2} n^{k+2} B A_n^{-1} \leq n^{-2} B_1.$$

Secondly, one might want to study the classes $F^p(A_n)$ ($0 < p < \infty$), where $F^p(A_n)$ is the set of functions $f(z) = \sum_0^\infty a_k z^k$ analytic in D such that $f^{(n)}$ has a continuous

extension to D^- ($n=0, 1, \dots$) and $\sum_1^\infty |a_k|^p A_k < \infty$. The following theorem shows, however, that the problem reduces to considering $F(A_n)$.

THEOREM 3.5. $F(n^{2/p} A_n^{1/p}) \subset F^p(A_n) \subset F(A_n^{1/p})$; and hence $F^p(A_n)$ is quasi-analytic if and only if (1) diverges.

The proof of this theorem is straightforward and is left to the reader.

4. An analogue in the half-plane. The class $F(A_n)$ has a "continuous analogue" in the following sense. Given a real-valued function $A(t)$ defined on the open interval $(0, \infty)$, let $F(A(t))$ be the set of all functions f analytic in the upper half-plane $H = \{z : \text{Im}(z) > 0\}$ such that

- (i) $f^{(n)}$ has a continuous extension to H^- ($n=0, 1, \dots$),
- (ii) f is of the form $f(z) = \int_0^\infty e^{itz} \hat{f}(t) dt$, where $\hat{f}(t)$ is a measurable function and $|\hat{f}(t)|A(t)$ is bounded.

We call $F(A(t))$ quasi-analytic if $f \in F(A(t))$ and $f^{(n)}(0) = 0$ ($n=0, 1, \dots$) together imply $f \equiv 0$.

Thus, looking at the boundary values of functions in $F(A(t))$ and $F(A_n)$, functions in $F(A(t))$ have Fourier transforms which vanish on the negative real line and go rapidly to zero on the positive real line, while the negative Fourier coefficients of functions in $F(A_n)$ vanish and the positive Fourier coefficients go rapidly to zero.

We restrict our attention to functions $A(t)$ for which $\log A(t)$ is a convex function of $\log t$, $A(t)$ is bounded below by 1, and $A(t)$ is bounded on the interval $(0, 1]$. Such a function has the following properties:

- (i) For $t_0 < t_1 < t_2$,

$$\begin{aligned} \frac{\log A(t_1) - \log A(t_0)}{\log t_1 - \log t_0} &\leq \frac{\log A(t_2) - \log A(t_0)}{\log t_2 - \log t_0} \\ &\leq \frac{\log A(t_2) - \log A(t_1)}{\log t_2 - \log t_1}. \end{aligned}$$

- (ii) For $h > 0$,

$$f(t; h) = \frac{\log A(t+h) - \log A(t)}{\log(t+h) - \log t}$$

is a decreasing function of h .

- (iii) $s(t) = \lim_{h \rightarrow 0^+} f(t; h)$ exists and is a nondecreasing function of t .
- (iv) For $t < t_0$,

$$\frac{\log A(t_0) - \log A(t)}{\log t_0 - \log t} \leq s(t_0).$$

Properties (i) and (ii) are statements about slopes of chords of the graph of $\log A(t)$ vs. $\log t$ and are clear when viewed geometrically. The function $s(t)$ measures the slope of the graph at $(\log t, \log A(t))$. The corresponding properties for convex functions are known [3, §3.18], so we omit a proof of the above properties.

LEMMA 4.1. *If $\lim_{t \rightarrow \infty} s(t) = \infty$, then for each nonnegative integer k , $t^k A(t)^{-1}$ is bounded.*

PROPOSITION 4.2. *Suppose $s(t) \geq 0$ and $\lim_{t \rightarrow \infty} s(t) = \infty$. Let $c(n) = \sup_{t \geq 1} t^n A(t)^{-1}$ and $d(t) = \sup_{n \geq 0} t^n c(n)^{-1}$. Then for $t \geq 1$, $d(t) \leq A(t) \leq t d(t)$. Furthermore, given n_0 there exists a number t_{n_0} such that for $t \geq t_{n_0}$, $d(t) = \sup_{n \geq n_0} t^n c(n)^{-1}$.*

The proofs of the lemma and proposition follow the corresponding proofs given in the previous section. It is interesting to note that if we let $c(r) = \sup_{t \geq 1} t^r A(t)^{-1}$ ($0 \leq r < \infty$) and $d(t) = \sup_{r \geq 0} t^r c(r)^{-1}$, then $d(t) = A(t)$.

THEOREM 4.3. *Suppose that $s(t) \geq 0$ and $\lim_{t \rightarrow \infty} s(t) = \infty$. Then a necessary and sufficient condition for $F(A(t))$ to be quasi-analytic is that*

$$(2) \quad \int_1^\infty t^{-3/2} \log A(t) dt$$

diverge.

Proof. Sufficiency. Suppose that f is in $F(A(t))$ and $f^{(n)}(0) = 0, 1, \dots$. Since $t^{n+2} A(t)^{-1} \leq c(n+2)$ for $t \geq 1$, the integral

$$f^{(n)}(z) = \int_0^\infty (it)^n e^{itz} \hat{f}(t) dt$$

converges uniformly for $\text{Im}(z) \geq 0$. In fact,

$$\begin{aligned} |f^{(n)}(z)| &\leq \int_0^\infty t^n B A(t)^{-1} dt \\ (3) \quad &\leq B + \int_1^\infty t^{-2} t^{n+2} B A(t)^{-1} dt \\ &\leq B + B c(n+2) \int_1^\infty t^{-2} dt \end{aligned}$$

for $n=0, 1, \dots$. If we let

$$T(r) = \sup_{n \geq 0} r^n c(n+2)^{-1} = r^{-2} \sup_{n \geq 2} r^{n+2} c(n+2)^{-1},$$

then $T(r) = r^{-2} d(r) \geq r^{-3} A(r)$ for $r \geq t_2$. Hence the divergence of (2) implies that $\int_1^\infty r^{-3/2} \log T(r) dr$ diverges. Hence by Theorem 3 of [3], $f \equiv 0$.

Necessity. Let $M_0 = 1$ and $M_n = e^{-n} c(n-1)$, $n=1, 2, \dots$. As in the proof of Theorem 3.3, the convergence of (2) implies that $\int_1^\infty r^{-3/2} \log T(r) dr$ diverges, where $T(r) = \sup_{n \geq 0} r^n M_n^{-1}$. Hence by Theorem 3 of [3] there is a function $f \neq 0$ analytic in the upper half-plane H such that $f^{(n)}$ has a continuous extension to H^- , $|f^{(n)}| \leq M_n$ and $f^{(n)}(0) = 0$ for $n=0, 1, \dots$. Let $g(z) = (z+i)^{-2} f(z)$ then g has the same properties as f . The proof will be complete if we show that g is in $F(A(t))$.

$$g^{(n)}(z) = \sum_{k=0}^n C_{n,k} f^{(n-k)}(z) (z+i)^{-(2+k)} (-1)^k (k+1)!.$$

By Lemma 2.1

$$f^{(n-k)}(z) = \frac{1}{(k-1)!} \int_0^z (z-w)^{k-1} f^{(n)}(w) dw.$$

Hence

$$|f^{(n-k)}(z)| \leq \frac{1}{(k-1)!} z^k B M_n, \quad k = 1, 2, \dots$$

And

$$|g^{(n)}(z)| \leq C_{n,0} |f^{(n)}(z)| |z+i|^{-2} + \left[\sum_{k=1}^n C_{n,k} \frac{1}{(k-1)!} |z|^k B M_n \right] [|z+i|^{-(2+k)} (k+1)!].$$

Since the second term is less than

$$\sum_{k=1}^n C_{n,k} B M_n |z+i|^{-2} (k+1)k,$$

we have

$$|g^{(n)}(z)| \leq B_1 e^n M_n |z+i|^{-2}.$$

In particular, g is in the class H^2 of the upper half-plane. Therefore by the Paley-Wiener theorem, the Fourier transform $\hat{g}(t)$ vanishes for $t < 0$. For $t > 0$,

$$2\pi \hat{g}(t) = \int_{-\infty}^{\infty} e^{-itx} g(x) dx = \int_{-\infty}^{\infty} (-it)^{-n} g^{(n)}(x) e^{itx} dx.$$

So that

$$|2\pi \hat{g}(t)| \leq t^{-n} B e^n M_n \int_{-\infty}^{\infty} |x+i|^{-2} dx.$$

Hence for $t \geq 1$,

$$\begin{aligned} \hat{g}(t) &\leq B_1 \inf_{n \geq 1} t^{-n} e^n M_n = B_1 t^{-1} \inf_{n \geq 0} t^{-n} c(n) \\ &\leq B_1 t^{-1} d(t)^{-1} \leq B_2 A(t)^{-1}. \end{aligned}$$

And for $0 < t < 1$,

$$|\hat{g}(t)| \leq B_1 M_0 \leq B_2 A(t)^{-1},$$

since $A(t)$ is bounded on the interval $(0, 1)$. Thus g is in $F(A(t))$.

As before, we immediately improve Theorem 4.3 to obtain the main result.

THEOREM 4.4. *Suppose that $A(t)$ is defined and bounded below by 1 on the interval $[0, \infty)$. Suppose further that $A(t)$ is bounded on $(0, 1)$ and that $\log A(t)$ is a convex function of $\log t$. Then a necessary and sufficient condition for $F(A(t))$ to be quasi-analytic is that*

$$(2) \quad \int_1^{\infty} t^{-3/2} \log A(t) dt$$

diverge.

Proof. If $\lim_{t \rightarrow \infty} s(t) = \infty$, the proof proceeds as in the previous section. If not, then let $a = \sup_{0 \leq t \leq e} A(t)$ and choose b so that $b \geq s(t)$ for all t . Let

$$\begin{aligned}\bar{A}(t) &= at^{b \log t} \quad \text{for } t \geq e, \\ &= ae^b \quad \text{for } t \leq e.\end{aligned}$$

Then as in the previous section $F(\bar{A}(t))$ is contained in $F(A(t))$ and $F(\bar{A}(t))$ is not quasi-analytic. Hence $F(A(t))$ is not quasi-analytic. On the other hand (2) converges.

We again observe that if (2) diverges, then any function f analytic in the upper half-plane H and of the form $f(z) = \int_0^\infty \hat{f}(t)e^{itz} dt$ where $\hat{f}(t)A(t)$ is bounded, is already in $F(A(t))$. For the divergence of (2) implies that $\lim_{t \rightarrow \infty} s(t) = \infty$ and hence (3) shows that

$$f^{(n)}(z) = \int_0^\infty \hat{f}(t)(it)^n e^{itz} dt$$

converges uniformly on H^- . Hence $f^{(n)}(z)$ has a continuous extension to H^- .

5. Functions of several variables. Since we will be working in the space of n complex variables C^n , we wish to make use of multi-indices of nonnegative integers $m = (m_1, \dots, m_n)$. If m and l are multi-indices, r is an n -tuple of real or complex numbers, and a and b are real numbers, then

$$\begin{aligned}|m| &= m_1 + m_2 + \dots + m_n; \\ m + a &= (m_1 + a, m_2 + a, \dots, m_n + a); \\ r^m &= \prod_{j=1}^n r_j^{m_j}; \\ (m + a)^b &= \prod_{j=1}^n (m_j + a)^b; \quad \text{and} \\ m \leq l &\text{ means } m_j \leq l_j, j = 1, 2, \dots, n.\end{aligned}$$

We also make use of the following notation: if f is a function of $r = (r_1, \dots, r_n)$, then

$$f^{(m)}(r) = \frac{\partial^{|m|}}{\partial r_1^{m_1} \dots \partial r_n^{m_n}} f(r),$$

and

$$\int f(r) dr = \int f(r) dr_1 dr_2 \dots dr_n.$$

THEOREM 5.1. Suppose that $f = \sum_m a_m z^m$ is analytic in the polydisc

$$D = \{z \in C^n : |z_j| < 1; j = 1, \dots, n\},$$

that $\{\sum_{|m|=k} |a_m|^2\}^{1/2} \leq A_k$ for z in D ($k=0, 1, \dots$), and that for each multi-index m

(i) $f^{(m)}$ has a continuous extension to D^- ;

(ii) $f^{(m)}(1, \dots, 1) = 0$.

Then a necessary and sufficient condition to conclude that $f=0$ is that

$$(4) \quad \sum_{k=1}^{\infty} k^{-3/2} \log A_k$$

diverge.

The following two lemmas contain some of the computations used repeatedly in the proof of Theorem 5.1.

LEMMA 5.2. If $\sum_{|m| \geq k} \sum_{l \leq m} x(m, l)$ is a convergent series of nonnegative terms, then

$$\sum_{|l| \leq k} \sum_{m \geq l} x(m, l) \leq \sum_{|m| \geq k} \sum_{l \leq m} x(m, l).$$

Proof.

$$\begin{aligned} \sum_{|l|=k} \sum_{m \geq l} x(m, l) &= \sum_m \sum_{|l|=k, l \leq m} x(m, l) \\ &= \sum_{|m| \geq k} \sum_{|l|=k, l \leq m} x(m, l) \leq \sum_{|m| \geq k} \sum_{l \leq m} x(m, l). \end{aligned}$$

LEMMA 5.3. If $\sum_{|m|=j} |a_m|^2 \leq A_j^{-2}$ ($j=1, 2, \dots$) and (4) diverges, then for $k=1, 2, \dots$ and $b \geq 1/n$

$$\sum_{j=k}^{\infty} \sum_{|m|=j} |a_m|^2 (m+1)^b \leq ck^{nb+2} A_k^{-2}$$

for some constant c .

Proof. If $|m|=j \geq 1$, $(m+1)^{1/n} \leq (j+n)/n$ by the inequality of the arithmetic and geometric means. Hence $(m+1)^b \leq (j/n+1)^{nb} \leq (2j)^{nb}$ and

$$\sum_{|m|=j} |a_m|^2 (m+1)^b \leq (2j)^{nb} \sum_{|m|=j} |a_m|^2 \leq (2j)^{nb} A_j^{-2}.$$

Thus it suffices to prove that

$$\sum_{j=k}^{\infty} j^{nb} A_j^{-2} \leq ck^{nb+2} A_k^{-2} \quad (k=1, 2, \dots).$$

Since (4) diverges, it follows that $\lim_{k \rightarrow \infty} s_k = \infty$ (as in the proof of Theorem 3.4). Hence there exists an integer $k_0 \geq 1$ such that for $k \geq k_0$, $s_k \geq \frac{1}{2}nb + 1$. So for $j > k \geq k_0$,

$$\frac{\log A_j - \log A_k}{\log j - \log k} \geq s_k \geq \frac{1}{2}nb + 1.$$

And $A_k A_j^{-1} \leq (kj^{-1})^{(nb+2)/2}$. So that

$$\begin{aligned} \sum_{j=k}^{\infty} j^{nb} A_j^{-2} &= A_k^{-2} \left(k^{nb} + \sum_{j=k+1}^{\infty} A_k^2 A_j^{-2} j^{nb} \right) \\ &\leq A_k^{-2} k^{nb+2} \sum_{j=1}^{\infty} j^{-2}. \end{aligned}$$

This proves the lemma for $k \geq k_0$. By choosing c large enough we see that the lemma is true for $k \geq 1$.

Proof of Theorem 5.1. First suppose that (4) converges. Then by Theorem 3.4 there is a function $g(w) \neq 0$ of one complex variable such that g is in $F(A_k)$ and $g^{(k)}(1) = 0$ for $k = 0, 1, \dots$. Let $f(z) = g(z_1)$ for $z = (z_1, \dots, z_n)$ in the polydisc D . Then $\{\sum_{|m|=k} |a_m|^2\}^{1/2} \leq A_k$ for z in D ($k = 0, 1, \dots$) and f satisfies (i) and (ii) but $f \neq 0$.

Now suppose that (4) diverges. We wish to show that $f \equiv 0$. Notice that since f is analytic in each variable separately, it is enough to show that $f(r) = 0$ for all $r = (r_1, \dots, r_n)$ with $0 < r_j < 1$, $j = 1, \dots, n$. Fix such an r and define a function of one complex variable by

$$\phi(w) = ((1-r_1)w + r_1, \dots, (1-r_n)w + r_n).$$

Then ϕ maps the open unit disc into D (and the closed unit disc into D^-), $\phi(1) = (1, \dots, 1)$ and $\phi(0) = r$. Let

$$g(w) = f(\phi(w)) = \sum_{k=0}^{\infty} b_k w^k.$$

Then g is analytic in the open unit disc,

$$g^{(k)}(w) = \sum_{|m|=k} f^{(m)}(\phi(w)) r^m$$

has a continuous extension to D^- and $g^{(k)}(1) = 0$ for $k = 0, 1, \dots$. We will show later that $|b_k| \leq ck^{n+1} A_k^{-1}$ for $k = 0, 1, \dots$. Assuming that this has been done, we let $\bar{A}_k = A_k k^{-(n+1)}$, $k = 1, 2, \dots$. Then $\{b_k \bar{A}_k\}$ is bounded and g is in $F(\bar{A}_k)$. Now $\log \bar{A}_k = \log A_k - (n+1) \log k$ is convex in $\log k$ and $\sum_{k=1}^{\infty} k^{-3/2} \log \bar{A}_k$ diverges since (4) diverges. It follows that $F(\bar{A}_k)$ is quasi-analytic and $g \equiv 0$. In particular $f(r) = f(\phi(0)) = g(0) = 0$.

Thus it suffices to show that $|b_k| \leq ck^{n+1} A_k^{-1}$. To simplify notation, let

$$x(l, m) = \prod_{j=1}^n C_{m_j, l_j} (1-r_j)^{l_j} r_j^{m_j - l_j}.$$

Notice that $\sum_{l \leq m} x(l, m) = 1$. Furthermore

$$\begin{aligned} g(w) &= \sum_m a_m \sum_{l \leq m} x(l, m) w^{|l|} = \sum_l \sum_{m \geq l} a_m x(l, m) w^{|l|} \\ &= \sum_{k=0}^{\infty} \sum_{|l|=k} \sum_{m \geq l} a_m x(l, m) w^k. \end{aligned}$$

Thus we conclude that

$$b_k = \sum_{|l|=k} \sum_{m \geq l} a_m x(l, m).$$

Finally

$$|b_k| \leq \left\{ \sum_{|l|=k} \sum_{m \geq l} x(l, m) (m+1)^{-2} \right\}^{1/2} \left\{ \sum_{|l|=k} \sum_{m \geq l} |a_m|^2 x(l, m) (m+1)^2 \right\}^{1/2}.$$

So by Lemma 5.2

$$\begin{aligned} |b_k| &\leq \left\{ \sum_{|m| \geq k} \sum_{l \leq m} x(l, m)(m+1)^{-2} \right\}^{1/2} \left\{ \sum_{|m| \geq k} \sum_{l \leq m} |a_m|^2 x(l, m)(m+1)^2 \right\}^{1/2} \\ &= \left\{ \sum_{|m| \geq k} (m+1)^{-2} \right\}^{1/2} \left\{ \sum_{|m| \geq k} |a_m|^2 (m+1)^2 \right\}^{1/2} \\ &\leq c \left\{ \sum_{j=k}^{\infty} \sum_{|m|=j} |a_m|^2 (m+1)^2 \right\}^{1/2}. \end{aligned}$$

And by Lemma 5.3

$$|b_k| \leq c_1 \{k^{2n+2} A_k^{-2}\}^{1/2} = c_1 k^{n+1} A_k^{-1}.$$

We conclude this paper by observing that, as in the one variable case, if $f(z) = \sum_m a_m z^m$ is analytic in D , $\sum_{|m|=k} |a_m|^2 \leq A_k^{-2}$ ($k=1, 2, \dots$) and (4) diverges, then $f^{(l)}$ already has a continuous extension to D^- for all l . To see this

$$(5) \quad f^{(l)}(z) = \sum_{m \geq l} \frac{m!}{(m-l)!} a_m z^{m-l}$$

and

$$\sum_{m \geq l} \frac{m!}{(m-l)!} |a_m| \leq \sum_m |a_m| (m+1)^l$$

which is finite by computations similar to those above. Thus (5) converges uniformly on D^- .

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